Noncommutative spectral geometry and the deformed Hopf algebra structure of quantum field theory

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Abstract. We report the results obtained in the study of Alain Connes noncommutative spectral geometry construction focusing on its essential ingredient of the algebra doubling. We show that such a two-sheeted structure is related with the gauge structure of the theory, its dissipative character and carries in itself the seeds of quantization. From the algebraic point of view, the algebra doubling process has the same structure of the deformed Hops algebra structure which characterizes quantum field theory.

1. Noncommutative spectral geometry and the standard model

One may assume that near the Planck energy scale, the geometry of space-time ceases to have the simple continuous form we are familiar with. At high enough energy scales, quantum gravity effects turn on and they alter space-time. One can thus assume that at high energy scales, space-time becomes discrete and the coordinates do not longer commute. Such an approach could *a priori* be tested by its phenomenological and cosmological consequences.

Combining noncommutative geometry [1, 2] with the spectral action principle, led to Noncommutative Spectral Geometry (NCSG), used by Connes and collaborators [3] in an attempt to provide a purely geometric explanation for the Standard Model (SM) of electroweak and strong interactions. In their approach, the SM is considered as a phenomenological model, which dictates the geometry of space-time so that the Maxwell-Dirac action functional leads to

the SM action. The model is constructed to hold at high energy scales, namely at unification scale; to get its low energy consequences which will then be tested against current data, one uses standard renormalization techniques. Since the model lives at high energy scales, it can be used to investigate early universe cosmology [4]-[11] [1]. The purpose of this contribution is twofold: firstly, to investigate the physical meaning of the choice of the *almost* commutative geometry and its relation to quantization [12], and secondly to explore the relation of NCSG with the gauge structure of the theory and with dissipation [12]. We will show that Connes construction is intimately related with the deformed Hopf algebra characterizing quantum field theory (QFT) [13] and therefore the seeds of quantization are built in such a NCSG construction.

We start by summarizing the main ingredients of NCSG, composed by a two-sheeted space, made from the product of a four-dimensional smooth compact Riemannian manifold $\mathcal{M}$ with a fixed spin structure, by a discrete noncommutative space $\mathcal{F}$ composed by only two points. Thus, geometry is specified by the product of a continuous manifold for space-time times an internal geometry for the SM. The noncommutative nature of the discrete space $\mathcal{F}$ is denoted by a spectral triple $(\mathcal{A}, \mathcal{H}, D)$. The algebra $\mathcal{A} = C^\infty(\mathcal{M})$ of smooth functions on $\mathcal{M}$ is an involution of operators on the finite-dimensional Hilbert space $\mathcal{H}$ of Euclidean fermions; it acts on $\mathcal{H}$ by multiplication operators. The operator D is the Dirac operator $\partial\!\!\!/_\mathcal{M} = \sqrt{-1}\gamma^\mu \nabla^s_\mu$ on the spin Riemannian manifold $\mathcal{M}$; D is a self-adjoint unbounded operator in $\mathcal{H}$. The space $\mathcal{H}$ is the Hilbert space $L^2(\mathcal{M}, S)$ of square integrable spinors S on $\mathcal{M}$. Thus one obtains a model of pure gravity with an action that depends on the spectrum of the Dirac operator, which provides the (familiar) notion of a metric.

The most important ingredient in this NCSG model is the choice of the algebra constructed within the geometry of the two-sheeted space $\mathcal{M} \times \mathcal{F}$; it captures all information about space. The product geometry is specified by:

$$\mathcal{A} = \mathcal{A}_1 \otimes \mathcal{A}_2 , \qquad \mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 , \tag{1}$$

as a consequence of the noncommutative nature of the discrete space $\mathcal{F}$ composed by only two points. In other words, Eq. (1) expresses, at the algebraic level and at the level of the space of the states, the two-sheeted nature of the geometry space $\mathcal{M} \times \mathcal{F}$. Assuming $\mathcal{A}$ is symplectic-unitary, it can be written as [14]

$$\mathcal{A} = M_a(\mathbb{H}) \oplus M_k(\mathbb{C}) , \tag{2}$$

with $k = 2a$ and $\mathbb{H}$ being the algebra of quaternions, which encodes the noncommutativity of the manifold. The first possible value for the even number k is 2, corresponding to a Hilbert space of four fermions, but this choice is ruled out from the existence of quarks. The next possible value is $k = 4$ leading to the correct number of $k^2 = 16$ fermions in each of the three generations. Thus, we will consider the minimal choice that can accommodate the physics of the SM; certainly other choices leading to larger algebras which could accommodate particle beyond the SM sector could be possible.

The second basic ingredient of the NCSG is the spectral action principle stating that, within the context of the product $\mathcal{M} \times \mathcal{F}$, the bare bosonic Euclidean action is given by the trace of the heat kernel associated with the square of the noncommutative Dirac operator and is simply

$$\mathrm{Tr}(f(D/\Lambda)) , \tag{3}$$

where f is a cut-off function and Λ fixes the energy scale. This action can be seen *à la* Wilson as the bare action at the mass scale Λ. The fermionic term can be included in the action functional by adding $(1/2)\langle J\psi, D\psi \rangle$, where J is the real structure on the spectral triple and ψ is a spinor in the Hilbert space $\mathcal{H}$ of the quarks and leptons.

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[1] See the contribution of M. Sakellariadou in the same conference.

Since we are considering a four-dimensional Riemannian geometry, the trace $\mathrm{Tr}(f(D/\Lambda))$ can be expressed perturbatively as [15]-[18]

$$\mathrm{Tr}(f(D/\Lambda)) \sim 2\Lambda^4 f_4 a_0 + 2\Lambda^2 f_2 a_2 + f_0 a_4 + \cdots + \Lambda^{-2k} f_{-2k} a_{4+2k} + \cdots , \qquad (4)$$

in terms of the geometrical Seeley-deWitt coefficients a_n, known for any second order elliptic differential operator. It is important to note that the smooth even test function f, which decays fast at infinity, appears through its momenta f_k:

$$\begin{aligned}
f_0 &\equiv f(0) , \\
f_k &\equiv \int_0^\infty f(u) u^{k-1} \mathrm{d}u , \quad \text{for} \ \ k > 0 , \\
\mathrm{f}_{-2k} &= (-1)^k \frac{k!}{(2k)!} f^{(2k)}(0) .
\end{aligned}$$

Since the Taylor expansion of the cut-off function vanishes at zero, the asymptotic expansion of Eq. (4) reduces to

$$\mathrm{Tr}(f(D/\Lambda)) \sim 2\Lambda^4 f_4 a_0 + 2\Lambda^2 f_2 a_2 + f_0 a_4 . \qquad (5)$$

Hence, the cut-off function f plays a rôle only through its three momenta f_0, f_2, f_4, which are three real parameters, related to the coupling constants at unification, the gravitational constant, and the cosmological constant, respectively. More precisely, the first term in Eq. (5) which is in in Λ^4 gives a cosmological term, the second one which is in Λ^2 gives the Einstein-Hilbert action functional, and the third one which is Λ-independent term yields the Yang-Mills action for the gauge fields corresponding to the internal degrees of freedom of the metric.

The NCSG offers a purely geometric approach to the SM of particle physics, where the fermions provide the Hilbert space of a spectral triple for the algebra and the bosons are obtained through inner fluctuations of the Dirac operator of the product $\mathcal{M} \times \mathcal{F}$ geometry. The computation of the asymptotic expression for the spectral action functional results to the full Lagrangian for the Standard Model minimally coupled to gravity, with neutrino mixing and Majorana mass terms. Supersymmetric extensions have been also considered.

In this report we closely follow the presentation of Ref. [12] (see also [19]). The relation between the algebra doubling and the deformed Hopf algebra structure of QFT are discussed in Section 2; dissipation, the gauge structure and quantization in Section 3. Section 4 is devoted to conclusions, where we also mention about the dissipative interference phase arising in the noncommutative plane in the presence of algebra doubling.

2. Noncommutative spectral geometry and quantum field theory

Our first observation is that the doubling of the algebra, $\mathcal{A} \to \mathcal{A}_1 \otimes \mathcal{A}_2$ acting on the "doubled" space $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$, which expresses the two sheeted nature of the NCSG (cf. Eq. (1)), is a key feature of quantum theories. As observed also by Alain Connes in Ref. [1], already in the early years of quantum mechanics (QM), in establishing the "matrix mechanics" Heisenberg has shown that noncommutative algebras governing physical quantities are at the origin of spectroscopic experiments and are linked to the discretization of the energy of the atomic levels and angular momentum. One can convince himself that this is the case by observing that in the density matrix formalism of QM the coordinate $x(t)$ of a quantum particle is split into two coordinates $x_+(t)$ (going forward in time) and $x_-(t)$ (going backward in time). The forward in time motion and the backward in time motion of the density matrix $W(x_+, x_-, t) \equiv \langle x_+|\rho(t)|x_-\rangle = \psi^*(x_-, t)\psi(x_+, t)$, where $x_\pm = x \pm y/2$, is described indeed by "two copies" of the Schrödinger equation, respectively:

$$i\hbar \frac{\partial \psi(x_+, t)}{\partial t} = H_+ \psi(x_+, t), \qquad -i\hbar \frac{\partial \psi^*(x_-, t)}{\partial t} = H_- \psi^*(x_-, t), \qquad (6)$$

which can be written as

$$i\hbar \frac{\partial \langle x_+|\rho(t)|x_-\rangle}{\partial t} = \hat{H} \ \langle x_+|\rho(t)|x_-\rangle, \tag{7}$$

where H is given in terms of the two Hamiltonian operators $H_\pm$ as

$$\hat{H} = H_+ - H_- \ . \tag{8}$$

The introduction of a doubled set of coordinates, $(x_\pm, p_\pm)$ (or (x, p_x) and (y, p_y)) and the use of the two copies of the Hamiltonian $H_\pm$ operating on the outer product of two Hilbert spaces $\mathcal{H}_+ \otimes \mathcal{H}_-$ thus show that the eigenvalues of $\hat{H}$ are directly the Bohr transition frequencies $h\nu_{nm} = E_n - E_m$, which are at the basis of the explanation of spectroscopic structure. We have observed elsewhere that the doubling of the algebra is implicit also in the theory of the Brownian motion of a quantum particle [20] (see also [12] and the references there quoted) and the doubled degrees of freedom are known [13] to allow quantum noise effect.

It is thus evident the connection with the two sheeted NCSG. Moreover, it has been shown [12] that as a consequence of the algebra doubling Eq. (1) the NCSG construction has an intrinsic gauge structure and is a thermal dissipative field theory. As we will discuss below, this suggests that Connes construction carries in itself the seeds of quantization, namely it is more than just a classical theory construction.

Let us start by discussing the simple case of the massless fermion and the $U(1)$ local gauge transformation group. We will see how in this case the doubling of the algebra is related to the gauge structure of the theory. Extension to the massive fermion case, the boson case and non-Abelian gauge transformation groups is possible [21, 22]. The system Lagrangian is

$$\hat{L} = L - \tilde{L} = -\overline{\psi}\gamma^\mu \partial_\mu \psi + \overline{\tilde{\psi}}\gamma^\mu \partial_\mu \tilde{\psi}. \tag{9}$$

The fermion tilde-field $\tilde{\psi}(x)$, which satisfies the usual fermionic anticommutation relations and anticommutes with the field $\psi(x)$, is a "copy" (with the same spectrum and couplings) of the ψ-system. We thus "double" the field algebra by introducing such a tilde-field $\tilde{\psi}(x)$.

For simplicity, no coupling term of the field $\psi(x)$ with $\tilde{\psi}(x)$ is assumed in $\hat{L}$. In the quantized theory, let $a_{\mathbf{k}}^\dagger$ and $\tilde{a}_{\mathbf{k}}^\dagger$ denote the creation operators associated to the quantum fields ψ and $\tilde{\psi}$, respectively (all quantum number indices are suppressed except momentum). The vacuum $|0(\theta)\rangle$ of the theory is:

$$|0(\theta)\rangle = \prod_k \left[\cos\theta_k + \sin\theta_k a_{\mathbf{k}}^\dagger \tilde{a}_{\mathbf{k}}^\dagger\right] |0\rangle, \tag{10}$$

namely a condensate of couples of $a_{\mathbf{k}}^\dagger$ and $\tilde{a}_{\mathbf{k}}^\dagger$ modes. Here $|0\rangle$ denotes the vacuum $|0,0\rangle \equiv |0\rangle \otimes |\tilde{0}\rangle$, with $|0\rangle$ and $|\tilde{0}\rangle$ the vacua annihilated by the annihilation operators $a_{\mathbf{k}}$ and $\tilde{a}_{\mathbf{k}}$, respectively. On the other hand, $|0(\theta)\rangle$ is the vacuum with respect to the fields $\psi(\theta; x)$ and $\tilde{\psi}(\theta; x)$ which are obtained by means of the Bogoliubov transformation:

$$\psi(\theta; x) \ = \ B^{-1}(\theta)\psi(x)B(\theta), \tag{11a}$$

$$\tilde{\psi}(\theta; x) \ = \ B^{-1}(\theta)\tilde{\psi}(x)B(\theta) \ , \tag{11b}$$

where $B(\theta) \equiv e^{-i\mathcal{G}}$, with the generator $\mathcal{G} = -i \sum_{\mathbf{k}} \theta_k (a_{\mathbf{k}}^\dagger \tilde{a}_{\mathbf{k}}^\dagger - a_{\mathbf{k}} \tilde{a}_{\mathbf{k}})$. For simplicity, θ is assumed to be independent of space-time. Extension to space-time dependent Bogoliubov transformations can be done [22]. $|0(\theta)\rangle$ is a $SU(2)$ generalized coherent state [23].

The Hamiltonian for the $\{\psi(x), \tilde{\psi}(x)\}$ system is $\hat{H} = H - \tilde{H}$ (to be compared with Eq. (8)), and is given by $\hat{H} = \sum_{\mathbf{k}} \hbar\omega_{\mathbf{k}}(a_{\mathbf{k}}^\dagger a_{\mathbf{k}} - \tilde{a}_{\mathbf{k}}^\dagger \tilde{a}_{\mathbf{k}})$. The θ-vacuum $|0(\theta)\rangle$ is the the zero eigenvalue

eigenstate of $\hat{H}$. The relation $[a_{\mathbf{k}}^{\dagger} a_{\mathbf{k}} - \tilde{a}_{\mathbf{k}}^{\dagger} \tilde{a}_{\mathbf{k}}]|0(\theta)\rangle = 0$, for any $\mathbf{k}$ (and any θ) characterizes the θ-vacuum structure and it is called the θ-vacuum condition.

The space of states $\hat{\mathcal{H}} = \mathcal{H} \otimes \tilde{\mathcal{H}}$ is constructed by repeated applications of creation operators of $\psi(\theta; x)$ and $\tilde{\psi}(\theta; x)$ on $|0(\theta)\rangle$ and is called the θ-representation $\{|0(\theta)\rangle\}$. θ-representations corresponding to different values of the θ parameter are unitarily inequivalent representations of the canonical anti-commutation relations in QFT [13]. The state $|0(\theta)\rangle$ is known to be a finite temperature state [13, 21, 22, 24, 25], namely one finds that θ is a temperature-dependent parameter. This tells us that the algebra doubling leads to a thermal field theory. The Bogoliubov transformations then induce transition through system phases at different temperatures.

We now consider the subspace $\mathcal{H}_{\theta c} \subset \{|0(\theta)\rangle\}$ made of all the states $|a\rangle_{\theta c}$, including $|0(\theta)\rangle$, such that the θ-state condition

$$[a_{\mathbf{k}}^{\dagger} a_{\mathbf{k}} - \tilde{a}_{\mathbf{k}}^{\dagger} \tilde{a}_{\mathbf{k}}]|a\rangle_{\theta c} = 0 , \qquad \text{for any} \quad \mathbf{k}, \tag{12}$$

holds in $\mathcal{H}_{\theta c}$ (note that Eq. (12) is similar to the Gupta-Bleuler condition defining the physical states in quantum electrodynamics (QED)). Let $\langle ... \rangle_{\theta c}$ denote matrix elements in $\mathcal{H}_{\theta c}$; we have

$$\langle j_\mu(x) \rangle_{\theta c} = \langle \tilde{j}_\mu(x) \rangle_{\theta c}, \tag{13}$$

where $j_\mu(x) = \overline{\psi} \gamma^\mu \psi$ and $\tilde{j}_\mu(x) = \overline{\tilde{\psi}} \gamma^\mu \tilde{\psi}$. Equalities between matrix elements in $\mathcal{H}_{\theta c}$, say $\langle A \rangle_{\theta c} = \langle B \rangle_{\theta c}$, are denoted by $A \cong B$ and we call them θ-w-equalities (θ-weak equalities). They are classical equalities since they are equalities among c-numbers. $\mathcal{H}_{\theta c}$ is invariant under the dynamics described by $\hat{H}$ (even in the general case in which interaction terms are present in $\hat{H}$ provided that the charge is conserved).

The key point is that, due to Eq. (13), the matrix elements in $\mathcal{H}_{\theta c}$ of the Lagrangian Eq. (9) are invariant under the simultaneous local gauge transformations of ψ and $\tilde{\psi}$ fields given by

$$\psi(x) \rightarrow \exp\left[ig\alpha(x)\right]\psi(x), \qquad \tilde{\psi}(x) \rightarrow \exp\left[ig\alpha(x)\right]\tilde{\psi}(x) , \tag{14}$$

i.e.,

$$\langle \hat{L} \rangle_{\theta c} \rightarrow \langle \hat{L}' \rangle_{\theta c} = \langle \hat{L} \rangle_{\theta c} , \quad \text{in} \quad \mathcal{H}_{\theta c}, \tag{15}$$

under the gauge transformations (14). The tilde term $\overline{\tilde{\psi}} \gamma^\mu \partial_\mu \tilde{\psi}$ thus plays a crucial rôle in the θ-w-gauge invariance of $\hat{L}$ under Eq. (14). Indeed it transforms in such a way to compensate the local gauge transformation of the ψ kinematical term, i.e.,

$$\overline{\tilde{\psi}}(x) \gamma^\mu \partial_\mu \tilde{\psi}(x) \rightarrow \overline{\tilde{\psi}}(x) \gamma^\mu \partial_\mu \tilde{\psi}(x) + g \partial^\mu \alpha(x) \tilde{j}_\mu(x). \tag{16}$$

This suggests to us to introduce the vector field A'_μ by

$$g j^{\bar{\mu}}(x) A'_{\bar{\mu}}(x) \cong \overline{\tilde{\psi}}(x) \gamma^{\bar{\mu}} \partial_{\bar{\mu}} \tilde{\psi}(x) , \quad \bar{\mu} = 0, 1, 2, 3. \tag{17}$$

Here and in the following, the bar over μ means no summation on repeated indices. Thus, the vector field A'_μ transforms as

$$A'_\mu(x) \rightarrow A'_\mu(x) + \partial_\mu \alpha(x), \tag{18}$$

when the transformations of Eq. (14) are implemented. In $\mathcal{H}_{\theta c}$, A'_μ can be then identified with the conventional U(1) gauge vector field and can be introduced it in the original Lagrangian through the usual coupling term $ig\overline{\psi} \gamma^\mu \psi A'_\mu$.

The position (17) does not change the θ-vacuum structure. Therefore, provided that one restricts himself/herself to matrix elements in $\mathcal{H}_{\theta c}$, matrix elements of physical observables,

which are solely functions of the $\psi(x)$ field, are not changed by the position (17). Our identification of A'_μ with the U(1) gauge vector field is also justified by the fact that observables turn out to be invariant under gauge transformations and the conservation laws derivable from $\hat{L}$, namely in the simple case of Eq. (9) the current conservation laws, $\partial^\mu j_\mu(x) = 0$ and $\partial^\mu \tilde{j}_\mu(x) = 0$, are also preserved as θ-w-equalities when Eq. (17) is adopted. Indeed, one obtains [21, 22] $\partial^\mu j_\mu(x) \cong 0$ and $\partial^\mu \tilde{j}_\mu(x) \cong 0$. One may also show that

$$\partial^\nu F'_{\mu\nu}(x) \cong -g j_\mu(x), \qquad \partial^\nu F'_{\mu\nu}(x) \cong -g \tilde{j}_\mu(x), \tag{19}$$

in $\mathcal{H}_{\theta c}$. In the the Lorentz gauge from Eq. (19) we also obtain the θ-w-relations $\partial^\mu A'_\mu(x) \cong 0$ and $\partial^2 A'_\mu(x) \cong g j_\mu(x)$.

In conclusion, the "doubled algebra" Lagrangian (9) for the field ψ and its "double" $\tilde{\psi}$ can be substituted in $\mathcal{H}_{\theta c}$ by:

$$\hat{L}_{\mathrm{g}} \cong -1/4 F'^{\mu\nu} F'_{\mu\nu} - \overline{\psi}\gamma^\mu \partial_\mu \psi + ig\overline{\psi}\gamma^\mu \psi A'_\mu \ , \quad \text{in } \mathcal{H}_{\theta c} \ , \tag{20}$$

where, remarkably, the tilde-kinematical term $\overline{\tilde{\psi}}\gamma^\mu \partial_\mu \tilde{\psi}$ is replaced, in a θ-w-sense, by the coupling term $ig\overline{\psi}\gamma^\mu \psi A'_\mu$ between the gauge field A'_μ and the matter field current $\overline{\psi}\gamma^\mu\psi$.

Finally, in the case an interaction term is present in the Lagrangian (9), $\hat{L}_{\mathrm{tot}} = \hat{L} + \hat{L}_I$, $\hat{L}_I = L_I - \tilde{L}_I$, the above conclusions still hold provided $\mathcal{H}_{\theta c}$ is an invariant subspace under the dynamics described by $\hat{L}_{\mathrm{tot}}$.

Our discussion has thus shown that the "doubling" of the field algebra introduces a gauge structure in the theory: Connes two sheeted geometric construction has intrinsic gauge properties.

The algebraic structure underlying the above discussion is recognized to be the one of the noncommutative q-deformed Hopf algebra [24]. We remark that the Hopf coproduct map $\mathcal{A} \to \mathcal{A} \otimes \mathbb{1} + \mathbb{1} \otimes \mathcal{A} \equiv \mathcal{A}_1 \otimes \mathcal{A}_2$ is nothing but the map presented in Eq. (1) which duplicates the algebra. On the other hand, it can be shown [24] that the Bogoliubov transformation of "angle" θ relating the fields $\psi(\theta; x)$ and $\tilde{\psi}(\theta; x)$ to $\psi(x)$ and $\tilde{\psi}(x)$, Eqs. (11), are obtained by convenient combinations of the q-deformed Hopf coproduct $\Delta a_q^\dagger = a_q^\dagger \otimes q^{1/2} + q^{-1/2} \otimes a_q^\dagger$, with $q \equiv q(\theta)$ the deformation parameters and $a_q^\dagger$ the creation operators in the q-deformed Hopf algebra [24]. These deformed coproduct maps are noncommutative. All of this signals a deep physical meaning of noncommutativity in the Connes construction since the deformation parameter is related to the condensate content of $|0(\theta)\rangle$ under the constrain imposed by the θ-state condition Eq. (12). Actually, such a state condition is a characterizing condition for the system physical states. The crucial point is that a characteristic feature of quantum field theory [13, 24] is that the deformation parameter *labels* the θ-representations $\{|0(\theta)\rangle\}$ and, as already mentioned, for $\theta \neq \theta'$, $\{|0(\theta)\rangle\}$ and $\{|0(\theta')\rangle\}$ are unitarily inequivalent representations of the canonical (anti-)commutation rules [13, 25]. In turn, the physical meaning of this is that an order parameter exists, which assumes different θ-dependent values in each of the representations. Thus, the q-deformed Hopf algebra structure of QFT induces the *foliation* of the whole Hilbert space into physically inequivalent subspaces. From our discussion we conclude that this is also the scenario which NCSG presents to us.

One more remark in this connection is that in the NCSG construction the derivative in the discrete direction is a finite difference quotient [1, 2, 12] and it is then suggestive that the q-derivative is also a finite difference derivative. This point deserves further formal analysis which is in our plans to do.

We thus conclude that Connes NCSG construction is built on the same noncommutative deformed Hopf algebra structure of QFT. In the next Section we show that it is also related with dissipation and carries in it the seeds of quantization.

3. Dissipation, gauge field and quantization

In the second equation in (19) the current $\tilde{j}_\mu$ act as the source of the variations of the gauge field tensor $F'_{\mu\nu}$. We express this by saying that the tilde field plays the rôle of a "reservoir". Such a reservoir interpretation may be extended also to the gauge field A'_μ, which is known to act, indeed, in a way to "compensate" the changes in the matter field configurations due to the local gauge freedom.

When we consider variations in the θ parameter (namely in the q-deformation parameter), induced by the Bogoliubov transformation generator, we have (time-)evolution over the manifold of the θ-labeled (i.e. q-labeled) spaces and we have dissipative fluxes between the doubled sets of fields, or, in other words, according to the above picture, between the system and the reservoir. We talk of dissipation and open systems when considering the Connes conctruction and the Standard Model in the same sense in a system of electromagnetically interacting matter field, neither the energy-momentum tensor of the matter field, nor that of the gauge field, are conserved. However, one verifies in a standard fashion [26] that $\partial_\mu T^{\mu\nu}_{\text{matter}} = eF^{\mu\nu}j_\mu = -\partial_\mu T^{\mu\nu}_{\text{gauge field}}$, so that what it is conserved is the *total* $T^{\mu\nu}_{\text{total}} = T^{\mu\nu}_{\text{matter}} + T^{\mu\nu}_{\text{gauge field}}$, namely the energy-momentum tensor of the *closed* system {matter field, electromagnetic field}. As remarked in Ref. [12], each element of the couple is *open* (dissipating) on the other one, although the *closeness* of the total system is ensured. Thus the closeness of the SM is not spoiled in our discussion.

In order to further clarify how the gauge structure is related to the algebra doubling and to dissipation we consider the prototype of a dissipative system, namely the classical one-dimensional damped harmonic oscillator

$$m\ddot{x} + \gamma\dot{x} + kx = 0 , \tag{21}$$

and its time-reversed $(\gamma \to -\gamma)$ (doubled) image

$$m\ddot{y} - \gamma\dot{y} + ky = 0 , \tag{22}$$

with time independent m, γ and k, needed [27] in order to set up the canonical formalism for open systems. The system of Eq. (21) and Eq. (22) is then a closed system described by the Lagrangian density

$$L(\dot{x}, \dot{y}, x, y) = m\dot{x}\dot{y} + \frac{\gamma}{2}(x\dot{y} - y\dot{x}) + kx\,y . \tag{23}$$

It is convenient to use the coordinates $x_1(t)$ and $x_2(t)$ defined by

$$x_1(t) = \frac{x(t) + y(t)}{\sqrt{2}} , \qquad x_2(t) = \frac{x(t) - y(t)}{\sqrt{2}} . \tag{24}$$

The motion equations are rewritten then as

$$m\ddot{x}_1 + \gamma\dot{x}_2 + kx_1 = 0 , \tag{25a}$$

$$m\ddot{x}_2 + \gamma\dot{x}_1 + kx_2 = 0 . \tag{25b}$$

The canonical momenta are: $p_1 = m\dot{x}_1 + (1/2)\gamma x_2$; $p_2 = -m\dot{x}_2 - (1/2)\gamma x_1$; the Hamiltonian is

$$\hat{H} = H_1 - H_2 = \frac{1}{2m}\left(p_1 - \frac{\gamma}{2}x_2\right)^2 + \frac{k}{2}x_1^2 - \frac{1}{2m}\left(p_2 + \frac{\gamma}{2}x_1\right)^2 - \frac{k}{2}x_2^2 . \tag{26}$$

We then recognize [28, 29, 21, 22] that

$$A_i = \frac{B}{2}\epsilon_{ij}x_j , \quad (i, j = 1, 2) , \qquad \epsilon_{ii} = 0 , \quad \epsilon_{12} = -\epsilon_{21} = 1 , \tag{27}$$

with $B \equiv c\gamma/e$, acts as the vector potential and obtain that the system of oscillators Eq. (21) and Eq. (22) is equivalent to the system of two particles with opposite charges $e_1 = -e_2 = e$ in the (oscillator) potential $\Phi \equiv (k/2/e)(x_1{}^2 - x_2{}^2) \equiv \Phi_1 - \Phi_2$ with $\Phi_i \equiv (k/2/e)x_i{}^2$ and in the constant magnetic field $\boldsymbol{B}$ defined as $\boldsymbol{B} = \nabla \times \boldsymbol{A} = -B\hat{\boldsymbol{3}}$. The Hamiltonian is indeed

$$\hat{H} = H_1 - H_2 = \frac{1}{2m}(p_1 - \frac{e_1}{c}A_1)^2 + e_1\Phi_1 - \frac{1}{2m}(p_2 + \frac{e_2}{c}A_2)^2 + e_2\Phi_2 \ . \tag{28}$$

and the Lagrangian of the system can be written in the familiar form

$$\begin{aligned} L &= \frac{1}{2m}(m\dot{x}_1 + \frac{e_1}{c}A_1)^2 - \frac{1}{2m}(m\dot{x}_2 + \frac{e_2}{c}A_2)^2 - \frac{e^2}{2mc^2}(A_1{}^2 + A_2{}^2) - e\Phi \\ &= \frac{m}{2}(\dot{x}_1^2 - \dot{x}_2^2) + \frac{e}{c}(\dot{x}_1 A_1 + \dot{x}_2 A_2) - e\Phi \ . \end{aligned} \tag{29}$$

Note the "minus" sign of the Lorentzian-like (pseudoeuclidean) metric in Eq. (29) (cf. also Eqs. (8), (9) and(28)), not imposed by hand, but derived through the doubling of the degrees of freedom and crucial in our description (and in the NCSG construction).

The doubled coordinate x_2 thus acts as the gauge field component A_1 to which the x_1 coordinate is coupled, and *vice versa*. The energy dissipated by one of the two systems is gained by the other one and viceversa, in analogy to what happens in standard electrodynamics as observed above. The picture is recovered of the gauge field as the bath or reservoir in which the system is embedded [21, 22].

Our toy system of harmonic oscillators Eq. (21) and Eq. (22) offers also an useful playground to show how dissipation is implicitly related to quantization. 't Hooft indeed has conjectured that classical, deterministic systems with loss of information might lead to a quantum evolution [30, 31, 32] provided some specific energy conditions are met and some constraints are imposed. Let us verify how such a conjecture is confirmed for our system described by Eq. (21) and Eq. (22). We will thus show that quantization is achieved as a consequence of dissipation. We rewrite the Hamiltonian Eq. (26) as $\hat{H} = H_{\mathrm{I}} - H_{\mathrm{II}}$, with

$$H_{\mathrm{I}} = \frac{1}{2\Omega\mathcal{C}}(2\Omega\mathcal{C} - \Gamma J_2)^2 \ , \quad H_{\mathrm{II}} = \frac{\Gamma^2}{2\Omega\mathcal{C}}J_2^2 \tag{30}$$

where C is the Casimir operator and J_2 is the (second) $SU(1,1)$ generator [29]:

$$\mathcal{C} = \frac{1}{4\Omega m}\left[\left(p_1^2 - p_2^2\right) + m^2\Omega^2\left(x_1^2 - x_2^2\right)\right] \ , \qquad J_2 = \frac{m}{2}\left[(\dot{x}_1 x_2 - \dot{x}_2 x_1) - \Gamma r^2\right] \ . \tag{31}$$

$\mathcal{C}$ is taken to be positive and

$$\Gamma = \frac{\gamma}{2m} \ , \ \Omega = \sqrt{\frac{1}{m}(k - \frac{\gamma^2}{4m})} \ , \ \text{with} \ \ k > \frac{\gamma^2}{4m} \ .$$

This $\hat{H}$ belongs to the class of Hamiltonians considered by 't Hooft and is of the form [33, 34]

$$\hat{H} = \sum_{i=1}^{2} p_i f_i(q) , \tag{32}$$

with $p_1 = \mathcal{C}$, $p_2 = J_2$, $f_1(q) = 2\Omega$, $f_2(q) = -2\Gamma$. Note that $\{q_i, p_i\} = 1$ and the other Poisson brackets are vanishing. The $f_i(q)$ are nonsingular functions of q_i and the equations for the q's, $\dot{q}_i = \{q_i, H\} = f_i(q)$, are decoupled from the conjugate momenta p_i. A complete set of

observables, called *beables*, then exists, which Poisson commute at all times. Thus the system admits a deterministic description even when expressed in terms of operators acting on some functional space of states $|\psi\rangle$, such as the Hilbert space [31]. Note that this description in terms of operators and Hilbert space, does not imply *per se* quantization of the system. Physical states are defined to be those satisfying the condition $J_2|\psi\rangle = 0$. This guaranties that $\hat{H}$ is bounded from below and $\hat{H}|\psi\rangle = H_{\mathrm{I}}|\psi\rangle = 2\Omega\mathcal{C}|\psi\rangle$. H_{I} thus reduces to the Hamiltonian for the two-dimensional "isotropic" (or "radial") harmonic oscillator $\ddot{r} + \Omega^2 r = 0$. Indeed, putting $K \equiv m\Omega^2$, we obtain

$$\hat{H}|\psi\rangle = H_{\mathrm{I}}|\psi\rangle = \left(\frac{1}{2m}p_r^2 + \frac{K}{2}r^2\right)|\psi\rangle. \tag{33}$$

The physical states are invariant under time-reversal ($|\psi(t)\rangle = |\psi(-t)\rangle$) and periodical with period $\tau = 2\pi/\Omega$. $H_{\mathrm{I}} = 2\Omega\mathcal{C}$ has the spectrum $\mathcal{H}_{\mathrm{I}}^n = \hbar\Omega n$, $n = 0, \pm1, \pm2, ...$; since $\mathcal{C}$ has been chosen to be positive, only positive values of n are considered. Then one obtains

$$\frac{\langle\psi(\tau)|\hat{H}|\psi(\tau)\rangle}{\hbar}\tau - \phi = 2\pi n \quad , \quad n = 0, 1, 2, \dots .$$

Using $\tau = 2\pi/\Omega$ and $\phi = \alpha\pi$, where α is a real constant, leads to

$$\mathcal{H}_{\mathrm{I,eff}}^n \equiv \langle\psi_n(\tau)|\hat{H}|\psi_n(\tau)\rangle = \hbar\Omega\left(n + \frac{\alpha}{2}\right). \tag{34}$$

$\mathcal{H}_{\mathrm{I,eff}}^n$ gives the effective nth energy level of the system, namely the energy given by $\mathcal{H}_{\mathrm{I}}^n$ corrected by its interaction with the environment. We conclude that the dissipation term J_2 of the Hamiltonian is responsible for the zero point ($n = 0$) energy: $E_0 = (\hbar/2)\Omega\alpha$. In QM the zero point energy *is* the "signature" of quantization since it is formally due to the nonzero commutator of the canonically conjugate q and p operators. Our conclusion is thus that the (zero point) "quantum contribution" E_0 to the spectrum of physical states is due and signals the underlying dissipative dynamics: dissipation manifests itself as "quantization".

We remark that the "full Hamiltonian" Eq. (32) plays the rôle of the free energy $\mathcal{F}$, and $2\Gamma J_2$ represents the heat contribution in $\hat{H}$ (or $\mathcal{F}$). Indeed, using $S \equiv (2J_2/\hbar)$ and $U \equiv 2\Omega\mathcal{C}$ and the defining relation for temperature in thermodynamics: $\partial S/\partial U = 1/T$, with the Boltzmann constant $k_{\mathrm{B}} = 1$, from Eq. (32) we obtain $T = \hbar\Gamma$, i.e., provided that S is identified with the entropy, $\hbar\Gamma$ can be regarded as the temperature. Note that it is not surprising that $2J_2/\hbar$ behaves as the entropy since it controls the dissipative part of the dynamics (thus the irreversible loss of information). It is interesting to observe that this thermodynamical picture is also cofirmed by the results on the canonical quantization of open systems in quantum field theory [27].

On the basis of these results (confirming 't Hooft's conjecture) we have proved that the NCSG classical construction, due to its essential ingredient of the doubling of the algebra, is intrinsically related with dissipation and thus carries in itself the seeds of quantization [12].

4. Conclusions

In Ref. [35] it has been shown that a "dissipative interference phase" also appears in the evolution of dissipative systems. Such a feature exhibits one more consequence of the algebra doubling which plays so an important rôle in the Connes NCSG construction. In this report we have discussed how such a doubling process implies the gauge structure of the theory, its thermal characterization, its built in dissipation features and the consequent possibility to exhibit quantum evolution behaviors. Here we only mention that the doubling of the coordinates also provides a relation between dissipation and noncommutative geometry in the plane. We refer

to Ref.[12] for details with reference to Connes construction (see also [19]). We only recall that in the (x_+, x_-) plane, $x_\pm = x \pm y/2$ (cf. Section 2), the components of forward and backward in time velocity $v_\pm = \dot{x}_\pm$, given by

$$v_\pm = \frac{\partial H}{\partial p_\pm} = \pm \frac{1}{m} \left(p_\pm \mp \frac{\gamma}{2} x_\mp \right) \ , \tag{35}$$

do not commute: $[v_+, v_-] = i\hbar \, \gamma/m^2$. Thus it is impossible to fix these velocities v_+ and v_- as being identical [35]. Moreover, one may always introduce a set of canonical conjugate position coordinates (ξ_+, ξ_-) defined by $\xi_\pm = \mp L^2 K_\mp$, with $\hbar K_\pm = m v_\pm$, so that $[\xi_+, \xi_-] = i L^2$, which characterizes the noncommutative geometry in the plane (x_+, x_-). L denotes the geometric length scale in the plane.

One can show [35] that an Aharonov–Bohm-type phase interference can always be associated with the noncommutative (X, Y) plane where

$$[X, Y] = i L^2 \ . \tag{36}$$

Eq. (36) implies the uncertainty relation $(\Delta X)(\Delta Y) \geq L^2/2$, due to the zero point fluctuations in the coordinates. Consider now a particle moving in the plane along two paths starting and finishing at the same point, in a forward and in a backward direction, respectively, forming a closed loop. The phase interference ϑ may be be shown [35] to be given by $\vartheta = \mathcal{A}/L^2$, where $\mathcal{A}$ is the area enclosed in the closed loop formed by the paths and Eq. (36) in the noncommutative plane can be written as

$$[X, P_X] = i\hbar \quad \text{where} \quad P_X = \left(\frac{\hbar Y}{L^2} \right) \ . \tag{37}$$

The quantum phase interference between two alternative paths in the plane is thus determined by the length scale L and the area $\mathcal{A}$. In the dissipative case, it is $L^2 = \hbar/\gamma$, and the quantum dissipative phase interference $\vartheta = \mathcal{A}/L^2 = \mathcal{A}\gamma/\hbar$ is associated with the two paths in the noncommutative plane, provided $x_+ \neq x_-$. When doubling is absent, i.e., $x_+ = x_-$, the quantum effect disappear. It is indeed possible to show [20] that in such a classical limit case the doubled degree of freedom is associated with "unlikely processes" and it may thus be dropped in higher order terms in the perturbative expansion, as indeed it happens in Connes construction. At the Grand Unified Theories scale, when inflation took place, the effect of gauge fields is fairly shielded. However, since these higher order terms are the ones responsible for quantum corrections, the second sheet cannot be neglected if one does not want to preclude quantization effects, as it happens once the universe entered the radiation dominated era.

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